# Dynamical Systems
# Tutorial 5: Period 3 implies chaos

May 12, 2019

Today's tutorial covers some results on periodic orbits in discrete maps. Recall:

**Definition 1.** *Let $f(x)$ be some function. We say that a point $x_0$ in the domain of $f$ is a **periodic point** with **period** $n$ if*

$$f^n(x_0) = x_0$$

*where $f^n(x)$ is the result of applying the function $f$ $n$ times in a row to $x$*

**Definition 2.** *A periodic orbit of period $n$ is the orbit $\{x_i\}_{i=1}^n$ where $f(x_i) = x_{i+1}$ (and $f(x_n) = x_1$), $f^n(x_i) = x_i$ for all $i$, and for any $k$, $1 \leq k \leq n-1$, $f^k(x_i) \neq x_i$ (so $n$ is the smallest value for which the orbit is periodic).*

## 1 Some History

In 1964, Sharkovskii published the following:

**Definition 3.** *The **Sharkovskii ordering** on the natural numbers is the following ordering:*

$$\begin{aligned}
& 2^0\cdot 3 \succ 2^0\cdot 5 \succ 2^0\cdot 7 \succ 2^0\cdot 9 \succ \ldots \succ 2^1\cdot 3 \succ 2^1\cdot 5 \succ 2^1\cdot 7 \succ 2^1\cdot 9 \succ \ldots \\
\ldots \succ\ & 2^2\cdot 3 \succ 2^2\cdot 5 \succ 2^2\cdot 7 \succ 2^2\cdot 9 \succ \ldots \succ 2^3\cdot 3 \succ 2^3\cdot 5 \succ 2^3\cdot 7 \succ 2^3\cdot 9 \succ \ldots \\
\ldots & \\
\ldots \succ\ & 2^5 \succ 2^4 \succ 2^3 \succ 2^2 \succ 2^1 \succ 1.
\end{aligned}$$

That is, first list all odd numbers except one, following by 2 times the odds, $2^2$ times the odds, $2^3$ times the odds, etc. This exhausts all the natural numbers with the exception of the powers of two which are listed last, in decreasing order.

**Theorem 1.** *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Suppose $f$ has a periodic orbit of period $k$. If $k \succ l$ in the above ordering, then $f$ also has a periodic orbit of period $l$.*

This leads to some interesting observations:

1. If $f$ has a periodic orbit whose period is not a power of 2, then $f$ necessarily has infinitely many periodic orbits. Conversely, if $f$ has only finitely many periodic orbits, then they all necessarily have periods which are powers of 2.
2. Period 3 is the greatest period in the Sharkovskii ordering and implies the existence of all other periods.
3. The converse of Sharkovskii's theorem is also true - there are maps which have periodic points of period $p$ and no "higher" period points according to the Sharkovskii ordering.

In 1975, unaware of Sharkovskii's result, Yorke and Li published the following theorem:

**Theorem 2.** *(Period three implies chaos) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $f$ has a periodic point of period three. Then $f$ is chaotic.*

**Remark.** The term "chaotic" here is somewhat distinguishable from the definition to chaos which we saw in class (which is the widely accepted definition due to Devaney), and is sometimes referred to as "chaotic in the sense of Li and Yorke". It requires

1. the existence of periodic orbits with period $n$ for every $n$, and
2. the existence of an uncountably infinite set $S$ that is *scrambled*, where a pair of points $x$ and $y$ is called "scrambled" if as the map is applied repeatedly to the pair, they get closer together and later move apart and then get closer together and move apart, etc., so that they get arbitrarily close together without staying close together. A set $S$ is called a scrambled set if every pair of distinct points in $S$ is scrambled. Scrambling is a kind of mixing.

* The uncountable set of chaotic points may, however, be of measure zero (in which case the map is said to have unobservable nonperiodicity or unobservable chaos).

Interesting historic side note: Li and Yorke are the ones who coined the term *chaos*.

We will start by proving the following result of Li and Yorke

**Theorem 3.** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $f$ has a periodic point of period three. Then $f$ has periodic points of all other periods.*

which is a special case of Sharkovskii's theorem.

## 2 Period three implies all other periods

*Proof.* We start by stating two elementary observations.

*Observation 1:* If $I, J$ are closed intervals with $I \subseteq J$ and $f(I) \supseteq J$, then $f$ has a fixed point in $I$. This is a simple consequence of the Intermediate Value Theorem.

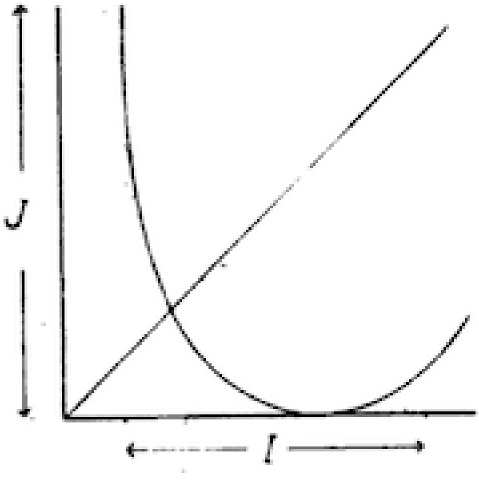

Fig. 10.1

*Observation 2:* suppose $A_0, A_1, ..., A_n$ are closed intervals and $f(A_i) \supseteq A_{i+1}$ for $i = 0, ..., n-1$. Then:

- there exists at least one subinterval $J_0$ of $A_0$ which is mapped onto $A_1$ (this is again a result of the IVT)

- There is a similar subinterval in $A_1$ which is mapped onto $A_2$, and thus there is a subinterval $J_1 \subseteq J_0$ having the property that $f(J_1) \subseteq A_1$ and $f^2(J_1) = A_2$.
- ... Continuing in this fashion, we find a nested sequence of intervals which map into the various $A_i$ in order.

Thus there exists a point $x \in A_0$ such that $f^i(x) \in A_i$ for each $i$. We say that $f(A_i)$ *covers* $A_{i+1}$.

Now, to prove the theorem, let $a,b,c \in \mathbb{R}$ $a < b < c$ a triple of points that form a 3-periodic orbit. Suppose that $f(a) = b, f(b) = c, f(c) = a$ without loss of generality, as the proof will proceed identically in the other case. Then we have $f(f(a)) = f(b) = c$.

Let $I_0 = [a,b]$ and $I_1 = [b,c]$. Note that because $f(a) = b, f(b) = c, f(c) = a$, by the intermediate value theorem, we have

- $f(I_0) = f([a,b]) \supseteq I_1 = [b,c]$, and
- $f(I_1) = f([b,c]) \supseteq I_0 \cup I_1 = [a,c]$.

Then, from our first observation, the graph of $f$ must have a fixed point for $f$ between $b$ and $c$. Similarly, for $f^2$:

$$f(f(I_0)) = f(f([a,b])) \supseteq [a,c], f(f(I_1)) = f(f([b,c])) \supseteq [b,c]$$

so $f^2$ must have fixed points, and at least one of these points must be of period 2. Let $n \geq 2$. We saw periodic points of periods 1,2 - our goal now is to produce a periodic point of period $n > 3$, for any $n$.

Inductively, we define a nested sequence of intervals $A_0, A_1, ..., A_{n-2} \subseteq I_1$ as follows.

- Set $A_0 = I_1$.
- Since $f(I_1) \supseteq I_1$, there is a subinterval $A_1 \subseteq A_0$ such that $f(A_1) = A_0 = I_1$.
- Then there is a subinterval $A_2 \subseteq A_1$ such that $f(A_2) = A_1$, so that $f^2(A_2) = A_0 = I_1$.
- Continuing, we find a subinterval $A_{n-2} \subseteq A_{n-3}$ such that $f(A_{n-2}) = A_{n-3}$. According to our second observation above, if $x \in A_{n-2}$, then

  $$f(x), f^2(x), ..., f^{n-2}(x) \subseteq A_0$$

  and indeed $f^{n-2}(A_{n-2}) = A_0 = I_1$.

Now, since $f(I_1) \supseteq I_0$, there exists a subinterval

$$A_{n-1} \subseteq A_{n-2}$$

such that

$$f^{n-1}(A_{n-1}) = I_0$$

Finally, since $f(I_0) \supseteq I_1$, we have $f^n(A_{n-1}) \supseteq I_1$ so that $f^n(A_{n-1})$ covers $A_{n-1}$. It follows then from our first observation that $f^n$ has a fixed point $p$ in $A_{n-1}$.

We claim that $p$ has period $n$. Indeed, the first $n-2$ iterations of $p$ lie in $I_1$, the $(n-1)$ iteration lies in $I_0$, and the $n-th$ is $p$ again. If $f^{n-1}(p)$ lies in the interior of $I_0$ then it follows easily that $p$ has period $n$. If $f^{n-1}(p)$ happens to lie on the boundary, then either $f^{n-1}(p) = a$ or $= b$, so $p = b$ or $p = c$ and $n = 3$. $\square$

# 3 Sketch of the proof to Sharkovskii's theorem

(Based the proof by Block, Guckenheimer, Misiurewicz and Young)

We introduce the following notation: for two closed intervals, $I_1$ and $I_2$, denote $I_1 \to I_2$ if $f(I_1)$ covers $I_2$ (i.e. $f(I_1) \supseteq I_2$). If we find a sequence of intervals $I_1 \to I_2 \to ... \to I_n \to I_1$, then our previous observations show that there is a fixed point of $f^n$ in $I_1$.

The idea of the proof is as follows. Assume $f$ has a periodic point $x$ of period $n$, with $n$ odd and $n > 1$, and assume $f$ has no periodic points of odd period less than $n$.

Let $x_1, ..., x_n$ be the points on the orbit of $x$ such that $x_1 < ... < x_n$. Clearly, $f$ permutes the $x_i$, $f(x_n) < x_n$ and $f(x_1) > x_1$. Let us choose the largest $i$ for which $f(x_i) > x_i$. Denote $I_1 = [x_i, x_{i+1}]$. Since $f(x_{i+1}) < x_{i+1}$ we have $f(x_{i+1}) \leq x_i$ and so $f(I_1) \supseteq I_1$. Therefore $I_1 \to I_1$.

On the other hand, since $x$ is not of period two, then $f(I_1)$ contains at least one other interval of the form $[x_j, x_{j+1}]$. Denote such an interval by $I_2$ - then $I_1 \to I_2$. We can now construct inductively a chain of intervals of the form $I_l = [x_j, x_{j+1}]$ such that $I_1 \to I_2 \to ... \to I_k$. Since there are only finitely many $x_j$, eventually there would be at least one interval $[x_j, x_{j+1}]$ whose image covers $I_1$. This follows since there are more $x_i$'s on one side of $I_1$ than on the other, hence some $x_i$ must change sides under the action of $f$, and some must not. Consequently there is at least one interval whose image covers $I_1$.

Now we have a chain $I_1 \to I_2 \to ... \to I_k \to I_1$, where each $I_l$ is of the form $[x_j, x_{j+1}]$ and $I_2 \neq I_1$. At least such chain exists. Let us choose the smallest $k$ for which this happens, i.e. this is the shortest path - see figure.

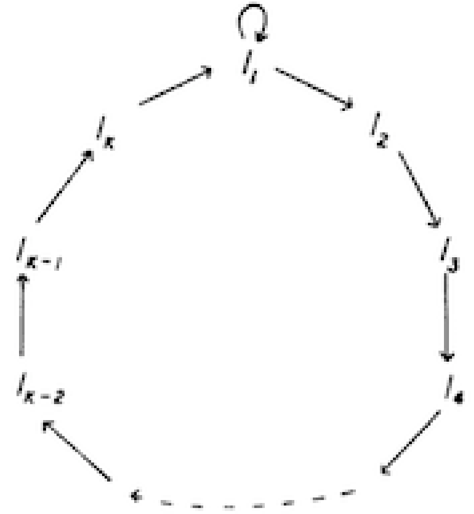

Fig. 10.3.

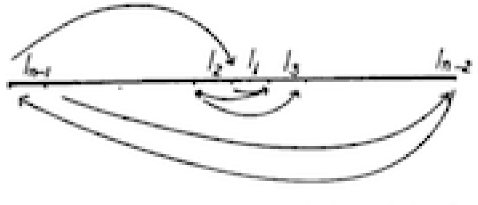

Fig. 10.4. One possible ordering of the $I_j$. The other is the mirror image.

If $k < n-1$, then either $I_1 \to I_2 \to ... \to I_k \to I_1$ or $I_1 \to I_2 \to ... \to I_k \to I_1 \to I_1$ gives a fixed point of $f^m$ with $m$ odd and $m < k$. This point must have period $< k$ (since $I_1 \cap I_2$ only consists of 1point with period $> m$). Therefore $k = n-1$.

Hence we cannot have $I_l \to I_j$ for any $j > l+1$ for any $j > l+1$. It follows (alternating points lemma) that the orbit of $x$ must be ordered in $\mathbb{R}$ in one of two possible ways, as depicted in the figure 10.4.

Hence we can extend the above diagram to the following (see next page):
and so construct:

- periods larger than $n$ by cycles of the form

$$I_1 \to ... \to I_{n-1} \to I_1 \to ...I_1$$

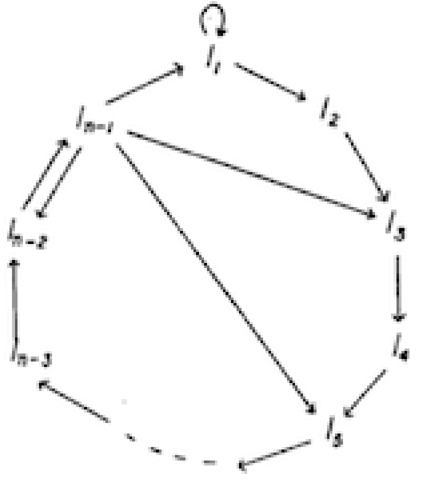

Fig. 10.5.

- smaller even periods by cycles of the form

$$I_{n-1} \rightarrow I_{n-2} \rightarrow I_{n-1},$$

$$I_{n-1} \rightarrow I_{n-4} \rightarrow I_{n-3} \rightarrow I_{n-2} \rightarrow I_{n-1}$$

  and so forth.

Completing the proof for $n$ even is done by using similar considerations and reductions to observations we have used above.

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## Other sources of interest

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